The Rokhlin Property for Automorphisms on Simple C^*-algebras

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Abstract

We study a general Kishimoto's problem for automorphisms on simple C^*-algebras with tracial rank zero. Let A be a unital separable simple C^*-algebra with tracial rank zero and let α be an automorphism. Under the assumption that α has certain Rokhlin property, we present a proof that $A \rtimes_\alpha \mathbb{Z}$ has tracial rank zero. We also show that if the induced map α_{*0} on $K_0(A)$ fixes a "dense" subgroup of $K_0(A)$ then the tracial Rokhlin property implies a stronger Rokhlin property. Consequently, the induced crossed product C^*-algebras have tracial rank zero.

1 Introduction

The Rokhlin property in ergodic theory was adopted to the context of von Neumann algebras by Connes ([3]). The Rokhlin property (with various versions) was also introduced to the study of automorphisms on C^*-algebras (see, for example, Herman and Ocneanu ([13]), Rørdam ([35], Kishimoto ([10]) and Phillips [34] among others– see also the next section).

A conjecture of Kishimoto can be formulated as follows: Let A be a unital simple A$\mathbb{T}$-algebra of real rank zero and α be an approximately inner automorphism. Suppose that α is "sufficiently outer", then the crossed product of the A$\mathbb{T}$-algebra by α, $A \rtimes_\alpha \mathbb{Z}$, is again a unital A$\mathbb{T}$-algebra. Kishimoto showed that this is true for a number of cases, in particular, for some cases that A has a unique tracial state.

Kishimoto proposed that the appropriate notion of outerness is the Rokhlin property ([11]). Kishimoto's problem has a more general setting:

P1 Let A be a unital separable simple C^*-algebra with tracial rank zero and let α be an automorphism. Suppose that α has a Rokhlin property. Does $A \rtimes_\alpha \mathbb{Z}$ have tracial rank zero?

If in addition A is assumed to be amenable and satisfy the Universal Coefficient Theorem, then, by the classification theorem ([20]), A is an AH-algebra with slow dimension growth and with real rank zero. If $A \rtimes_\alpha \mathbb{Z}$ has tracial rank zero, then, again, by [20], $A \rtimes_\alpha \mathbb{Z}$ is an AH-algebra (with slow dimension growth and with real rank zero). Note that simple A$\mathbb{T}$-algebras with real rank zero are exactly those simple AH-algebras with torsion free K-theory, with slow dimension growth and with real rank zero. It should also be noted ([18]) that a unital simple AH-algebra has slow dimension growth and real rank zero if and only if it has tracial rank zero.

If $K_i(A)$ are torsion free and α is approximately inner, then $K_i(A \rtimes_\alpha \mathbb{Z})$ is torsion free. Thus an affirmative answer to the problem **P1** proves the original Kishimoto's conjecture (see also [27]). One should also notice that if α is not approximately inner, $K_i(A \rtimes_\alpha \mathbb{Z})$ may have torsion even if A does not. Therefore, in Kishimoto's problem, the restriction that α is approximately inner can not be removed. So it is appropriate to replace $A\mathbb{T}$-algebras by AH-algebras if the requirement that α is approximately inner is removed.

In this paper we report some of the recent development on this subject.

Let A be a unital separable simple C^*-algebra with tracial rank zero and let α be an automorphism on A. In section 3, we present a proof that if α satisfies the tracial cyclic Rokhlin property (see 2.4 below) then the crossed product $A \rtimes_\alpha \mathbb{Z}$ has tracial rank zero which gives a

solution to **P1**, provided that $[\alpha^r] = [\mathrm{id}_A]$ in $KL(A, A)$. In section 4, we discuss when α has the tracial cyclic Rokhlin property. The tracial Rokhlin property introduced by N. C. Phillips (see [34] and [33]) has been proved to be a natural generalization of original Rokhlin towers for ergodic actions. We prove that, if in addition, $\alpha^r_{*0}|_G = \mathrm{id}|_G$ (for some integer $r \geq 1$) for some subgroup $G \subset K_0(A)$ for which $\rho_A(G)$ is dense in $\rho_A(K_0(A))$, then that α has tracial Rokhlin property implies that α has tracial cyclic Rokhlin property.

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2 Preliminaries

We will use the following convention:

(i) Let A be a C^*-algebra, let $a \in A$ be a positive element and let $p \in A$ be a projection. We write $[p] \leq [a]$ if there is a projection $q \in \overline{aAa}$ and a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$.

(ii) Let A be a C^*-algebra. We denote by $Aut(A)$ the automorphism group of A. If A is unital and $u \in A$ is a unitary, we denote by $\mathrm{ad}\, u$ the inner automorphism defined by $\mathrm{ad}\, u(a) = u^*au$ for all $a \in A$.

(iii) Let $T(A)$ be the tracial state space of a unital C^*-algebra A. It is a compact convex set. Denote by $Aff(T(A))$ the normed space of all real affine continuous functions on $T(A)$. Denote by $\rho_A : K_0(A) \to Aff(T(A))$ the homomorphism induced by $\rho_A([p])(\tau) = \tau(p)$ for $\tau \in T(A)$.

It should be noted, by [2], if A is a unital simple amenable C^*-algebra with real rank zero and stable rank and weakly unperforated $K_0(A)$, $\rho_A(K_0(A))$ is dense in $Aff(T(A))$.

(iv) Let A and B be two C^*-algebras and $\varphi, \psi : A \to B$ be two maps. Let $\epsilon > 0$ and $\mathcal{F} \subset A$ be a finite subset. We write

$$\varphi \approx_\epsilon \psi \text{ on } \mathcal{F},$$

if

$$\|\varphi(a) - \psi(a)\| < \epsilon \text{ for all } a \in \mathcal{F}.$$

(v) Let $x \in A$, $\epsilon > 0$ and $\mathcal{F} \subset A$. We write $x \in_\epsilon \mathcal{F}$, if $\mathrm{dist}(x, \mathcal{F}) < \epsilon$, or there is $y \in \mathcal{F}$ such that $\|x - y\| < \epsilon$.

(vi) If $h : A \to B$ is a homomorphism, then $h_{*i} : K_i(A) \to K_i(B)$ $(i = 0, 1)$ is the induced homomorphism.

We recall the definition of tracial topological rank of C*-algebras.

Definition 2.1. [15, Theorem 6.13] Let A be a unital simple C^*-algebra . Then A is said to have *tracial (topological) rank zero* if for any finite set $\mathcal{F} \subset A$, and $\epsilon > 0$ and any non-zero positive element $a \in A$, there exists a finite dimensional C^*-subalgebra $B \subset A$ with $\mathrm{id}_B = p$ such that

(1) $\|px - xp\| < \epsilon$ for all $x \in \mathcal{F}$,

(2) $pxp \in_\epsilon B$ for all $x \in \mathcal{F}$,

(3) $[1-p] \le [a]$.

If A has tracial rank zero, we write $\mathrm{TR}(A) = 0$.

If A is assumed to have the Fundamental Comparison Property (i.e., for any two projections $p, q \in A$, $\tau(p) < \tau(q)$ for all $\tau \in T(A)$ implies that p is equivalent to a projection $p' \le q$.), then the third condition may be replaced by $\tau(1-p) < \epsilon$ for all $\tau \in T(A)$. It is proved ([15], or see 3.7 of [17]) that if $TR(A) = 0$, then A has the Fundamental Comparison Property, as well as real rank zero and stable rank one. Every simple AH-algebra with real rank zero and with the Fundamental Comparison Property has tracial rank zero ([18] and [6]). Other simple C^*-algebras with tracial rank zero may be found in [21].

There are several versions of the Rokhlin property (see [13], [35], [10] and [7]).

The following is defined in [33, Definition 2.1].

Definition 2.2. Let A be a simple unital C^*-algebra and let $\alpha \in \mathrm{Aut}(A)$. We say α has the *tracial Rokhlin property* if for every finite set $\mathcal{F} \subset A$, every $\epsilon > 0$, every $n \in \mathbb{N}$, and every nonzero positive element $a \in A$, there are mutually orthogonal projections $e_1, e_2, \ldots, e_n \in A$ such that:

(1) $\|\alpha(e_j) - e_{j+1}\| < \epsilon$ for $1 \le j \le n-1$.

(2) $\|e_j a - a e_j\| < \epsilon$ for $0 \le j \le n$ and all $a \in \mathcal{F}$.

(3) With $e = \sum_{j=0}^{n} e_j$, $[1-e] \le [a]$.

The following result of Osaka and Phillips is the tracial Rokhlin version of Kishimoto's result in the case of simple unital $A\mathbb{T}$-algebras with a unique trace [11, Theorem 2.1].

Theorem 2.3. cf. [33] Let A be a simple unital C^*-algebra with $\mathrm{TR}(A) = 0$, and suppose that A has a unique tracial state. Then the following conditions are equivalent:

(1) α has the tracial Rokhlin property.

(2) α^m is not weakly inner in the GNS representation π_τ for any $m \neq 0$.

(3) $A \rtimes_\alpha \mathbb{Z}$ has real rank zero.

(4) $A \rtimes_\alpha \mathbb{Z}$ has a unique trace.

We define a stronger version of the tracial Rokhlin property similar to the approximately Rokhlin property in [11, Definition 4.2].

Definition 2.4. Let A be a simple unital C^*-algebra and let $\alpha \in \mathrm{Aut}(A)$. We say α has the *tracial cyclic Rokhlin property* if for every finite set $\mathcal{F} \subset A$, every $\epsilon > 0$, every $n \in \mathbb{N}$, and every nonzero positive element $a \in A$, there are mutually orthogonal projections $e_0, e_1, \ldots, e_n \in A$ such that

(1) $\|\alpha(e_j) - e_{j+1}\| < \epsilon$ for $0 \le j \le n$, where $e_{n+1} = e_0$.

(2) $\|e_j a - a e_j\| < \epsilon$ for $0 \le j \le n$ and all $a \in \mathcal{F}$.

(3) With $e = \sum_{j=0}^{n} e_j$, $[1-e] \le [a]$.

The following is a restatement of Theorem 3.4 of [22].

Theorem 2.5. *Let C be a unital AH-algebra and let A be a unital simple C^*-algebra with tracial rank zero. Suppose that $\varphi_1, \varphi_2 : C \to A$ are two unital monomorphisms such that*

$$[\varphi_1] = [\varphi_2] \text{ in } KL(C, A) \text{ and } \tau \circ \varphi_1 = \tau \circ \varphi_2 \textit{ for all } \tau \in T(A).$$

If also $K_1(A) = H_1(K_0(C), K_1(B))$ (see [26]), then there exists a sequence of unitaries $u_n \in U_0(A)$ such that

$$\lim_{n\to\infty} \mathrm{ad}\, u_n \circ \varphi_1(f) = \varphi_2(f) \textit{ for all } f \in C.$$

Proof. This follows from 3.6 of [23] that the above statement holds without requiring u_n in $U_0(A)$. The reason that u_n can be taken in $U_0(A)$ is given in [26]–see the proof of 12.4 of [25]. □

Lemma 2.6. *Let A be a unital simple C^*-algebra with stable rank one and let F be a finite dimensional C^*-subalgebra. Suppose that there are two monomorphisms $\varphi_1, \varphi_2 : F \to A$ such that φ_1 is not unital and*

$$(\varphi_1)_{*0} = (\varphi_2)_{*0}.$$

Then there exists a unitary $u \in U_0(A)$ such that

$$u^*\varphi_1(a)u = \varphi_2(a), \quad \text{for all } a \in F.$$

Proof. Since A has stable rank one, it is well known that there is a unitary $v \in U(A)$ such that

$$v^*\varphi_1(a)v = \varphi_2(a) \text{ for all } a \in F.$$

Since φ_1 is not unital and A is of stable rank one, neither is φ_2. Let $e = 1 - \varphi_2(1_F)$. Then $e \neq 0$.

Since A is simple, eAe is stably isomorphic to A. Furthermore, the map $w \mapsto (1 - q_1) + w$ is an isomorphism from $K_1(q_1Aq_1)$ onto $K_1(A)$. Therefore, since A has stable rank one, there exists $w \in U(eAe)$ such that $[(1 - e) + w] = [v]$ in $K_1(A)$. Define

$$u = v(1 - e + w)^*.$$

Then $u \in U_0(A)$. Moreover,

$$u^*\varphi_1(a)u = \varphi_2(a) \text{ for all } a \in F.$$

□

Definition 2.7. Let $f : S^1 \to S^1$ be a degree k map ($k > 1$), i.e., a continuous map with the winding number k. Following 4.2 of [6], denote by $T_{II,k} = D^2 \cup_f S^1$ the connected finite CW complex obtained by attaching a 2-cell D^2 to S^1 via the map f.

Let $g : S^2 \to S^2$ be a degree k ($k > 1$) map. Let $T_{III,k} = D^3 \cup_g S^2$ the connected finite CW complex obtained by attaching a 3-cell D^3 to S^2 via the map g.

Let $C = \oplus_{i=1}^r C_i$, where $C_i = P_iM_{k_i}(C(X_i))P_i$, where $P_i \in M_{r_i}(C(X_i))$ is a projection and X_i is a point, $X_i = S^1$, $X_i = T_{II,m_i}$ or T_{III,M_i}.

Lemma 2.8. *Let $C = \oplus_{i=1}^r C_i$ be a unital C^*-algebra, where C_i is as described in 2.7, Let $\epsilon > 0$, let $\mathcal{F} \subset C$ be a finite subset. There is $\delta > 0$ and a finite subset $\mathcal{G} \subset C$ satisfying the following. Suppose that A is a unital separable simple C^*-algebra with $TR(A) = 0$ and suppose that $\varphi : C \to A$ be a unital monomorphism and $u \in U(A)$ is a unitary such that*

$$\|[u, \varphi(g)]\| < \delta, \tag{e 2.1}$$

then there exists a unitary $v \in U(A)$ such that

$$\|[v, \varphi(f)]\| < \epsilon \textit{ for all } f \in \mathcal{F} \text{ and } \mathrm{Bott}(uv, \varphi) = 0, \tag{e 2.2}$$

$$\text{or } \mathrm{Bott}(vu, \varphi) = 0. \tag{e 2.3}$$

This is a combination of 6.7, 6.8, 6.9 and 6.10 of [25].

Lemma 2.9. *Let A be a unital simple C^*-algebra of tracial rank zero, let C be another C^*-algebra such that $A \subset C$ with $1_C = 1_A$, and let $B_1 = \oplus_{i=1}^r C_i$, where C_i is as described in 2.7 with $1_C = 1_A$. Let B_2 be a finite dimensional C^*-algebra with $p = 1_{B_2}$. Suppose that $px = xp$ for all $x \in B_1$, $x \mapsto (1-p)x(1-p)$ is injective and $pC_lp \neq 0$, $l = 1, 2, ..., r$. Let $B_3 = B_2 \oplus (1-p)B_1(1-p)$ and let $\varphi_0 : B_3 \to A$ be the embedding.*

Suppose that $U \in U(C)$ with $U^*aU \in A$, $\tau(U^*aU) = \tau(a)$ for all $a \in A$ and $\tau \in T(A)$ and

$$[\mathrm{ad}U \circ \varphi_0] = [\varphi_0] \text{ in } KL(A, A). \tag{e 2.4}$$

Let $\epsilon > 0$, $\mathcal{F} \subset B_1$ and let $m \geq 1$ be a finite subset. Then there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset B_1$ satisfying the following:

Suppose that $V \in U(A)$ with $V^*U^*1_{C_j}UV = 1_{C_j}$, $1 \leq j \leq r$, such that

$$\|V^*U^*aUV - a\| < \delta \text{ for all } a \in \mathcal{G} \tag{e 2.5}$$

and suppose that $V = V_1V_2 \cdots V_m$ for some $m \geq 2$ with

$$\|V_i - 1\| < \frac{\pi}{(m-1)}, \quad i = 1, 2, ..., m. \tag{e 2.6}$$

Then there exist unitaries $W_1, W_2, ..., W_m \in U(A)$ such that

$$\|W_i - 1\| < \frac{2\pi}{m-1} + \|V_i - 1\|, \quad i = 1, 2, ..., m-1, \tag{e 2.7}$$

$$\|(W_1W_2 \cdots W_l)b(W_1W_2 \cdots W_l)^* - (V_1V_2 \cdots V_l)b(V_1V_2 \cdots V_l)^*\| < \epsilon \tag{e 2.8}$$

for all $b \in \mathcal{F}$ and

$$(W_1W_2 \cdots W_m)^*U^*aU(W_1W_2 \cdots W_m) = a \tag{e 2.9}$$

for all $a \in B_2$.

Proof. Let $E(l)$ be the identity of C_l, $l = 1, 2, ..., r$. By the assumption,

$$V^*U^*E(l)UV = E(l), \quad l = 1, 2, ..., r. \tag{e 2.10}$$

Note that $pE(l) = E(l)p \neq 0$, $l = 1, 2, ..., r$. Therefore, by considering each summand individually, without loss of generality, we may assume that $r = 1$. So for the rest of the proof, $C_1 = B_1$. Denote by $\varphi : C_1 \to A$ the embedding and denote by $\varphi' : C_1 \to (1-p)A(1-p)$ and $\varphi'' : C_1 \to B_2 \subset pAp$ the homomorphisms defined by $\varphi'(c) = (1-p)\varphi(c)(1-p)$ and by $\varphi''(c) = pcp$ all $c \in C_1$, respectively.

Therefore one has $pB_1 = B_1p = M_{k(1)}$. Then $B_2 = M_{k_1}(F)$ for some finite dimensional C^*-algebra F. By 2.6, there exists a unitary $W' \in U_0(A)$ such that

$$(W')^*V^*U^*bUVW' = b \text{ for all } b \in B_2. \tag{e 2.11}$$

In particular, for all $c \in C_1 = B_1$,

$$(W')^*V^*U^*pcpUVW' = pcp, \tag{e 2.12}$$

Choose $\delta_1 > 0$ and a finite subset $\mathcal{G}_1 \subset B_1$ such that the following holds:

$$\mathrm{Bott}(Z, \psi_1) = \mathrm{Bott}(Z, \psi_2) \tag{e 2.13}$$

for any unitary $Z \in U(A)$ and any unital homomorphisms $\psi_1, \psi_2 : B_1 \to A$, whenever

$$\|\psi_1(f) - \psi_2(f)\| < \delta_1 \text{ for all } f \in \mathcal{G}_1,$$

provided that both sides in (e 2.13) are defined (see [24]).

Our δ and $\mathcal{G}$ will be chosen later, but, at least, we will choose $\delta < \delta_1$ and $\mathcal{G} \supset \mathcal{G}_1$.

For any given $\delta_2 > 0$ and finite subset $\mathcal{G}_2 \subset B_1$, let $\delta_3 > 0$ be as in 2.8 (in place of δ) and let $\mathcal{G}_3$ be a finite subset as in 2.8 (in place of $\mathcal{G}_2$) associated with $\delta/4$ (in place of ϵ) and $\mathcal{G}$ (in place of $\mathcal{F}$). By 3.4 of [22] (see also 3.6 of [23]), from the assumption (e 2.4) and the assumption that $\tau(U^*xU) = \tau(x)$ for all $x \in A$ and $\tau \in T(A)$, there is a unitary $U' \in U(A)$ such that

$$\|(U')^*cU' - U^*cU\| < \delta_3/2 < \delta_1/4 \text{ for all } c \in \mathcal{G}_3 \text{ and} \qquad (e\,2.14)$$

$$(U')^*bU' = U^*bU \text{ for all } b \in B_2. \qquad (e\,2.15)$$

Moreover, by applying 2.8, if $\|V^*U^*cUV - c\| < \delta_3/2$ for all $c \in \mathcal{G}_3$, we can obtain another unitary $U'' \in U(A)$ such that $(U'')^*bU'' = b$ for all $b \in B_2$,

$$\|U''c - cU''\| < \delta_2 \text{ for all } c \in \mathcal{G}_2 \text{ and } \mathrm{Bott}(VU'U'', \varphi) = 0. \qquad (e\,2.16)$$

By choosing $\delta < \delta_3$, $\mathcal{G} \supset \mathcal{G}_3$ and replacing U' by $U''U'$, simplifying the notation, we may simply assume (omitting δ_2 and $\mathcal{G}_2$) that

$$\mathrm{Bott}(VU', \varphi) = 0, \;\; \|(U')^*cU' - U^*cU\| < \delta/4 \text{ for all } c \in \mathcal{G} \qquad (e\,2.17)$$

$$\text{and } (U')^*bU' = U^*bU \text{ for all } b \in B_2 \qquad (e\,2.18)$$

By (e 2.4) and (e 2.12),

$$[\mathrm{ad}UVW' \circ \varphi'] = [\varphi']. \qquad (e\,2.19)$$

Then, by applying [22] (see also 3.6 of [23]), if $\delta > 0$ (we may assume that $\delta < \epsilon/4m$) and a finite subset $\mathcal{G} \subset B_1$ are given, there is a unitary $W'' \in (1-p)A(1-p)$ such that

$$\|(W'')^*(W')^*V^*U^*(1-p)c(1-p)UVW'W'' - (1-p)c(1-p)\| < \delta/8 \text{ for all } c \in \mathcal{G}. \qquad (e\,2.20)$$

It follows that

$$\|(W'')^*(W')^*V^*(U')^*(1-p)c(1-p)U'VW'W'' - (1-p)c(1-p)\| < 3\delta/8 \qquad (e\,2.21)$$

for all $c \in \mathcal{G}$.

For the monomorphism φ, we choose $\eta > 0$ and a finite subset $\mathcal{G}_2 \in B_1$ so that 17.5 (see also 8.4) of [24] can be applied for $\epsilon/2m$ (in place of ϵ) and $\mathcal{F}$. We may assume that $\mathcal{F} \subset \mathcal{G}_2$.

Then, by choosing large $\mathcal{G}$ and small δ, by 2.8, there is a unitary $W_0 \in U((1-p)A(1-p))$ such that

$$\|W_0(1-p)c(1-p) - (1-p)c(1-p)W_0\| < \eta/4 \text{ for all } \mathcal{G}_1 \text{ and} \qquad (e\,2.22)$$

$$\mathrm{Bott}(((1-p)U'VW'W''W_0), \varphi')) = 0. \qquad (e\,2.23)$$

Moreover, we may assume that $\delta < \eta/4$ and $\mathcal{G} \supset \mathcal{G}_2$. Keep in mind that we also assume that (e 2.5) holds for the above δ and $\mathcal{G}$. Put $W = W'(W'' \oplus p)(W_0 \oplus p)$. Then,

$$W^*V^*U^*cUV^*W = W^*V^*U^*(1-p)c(1-p)UVW + W^*V^*U^*qcqUVW \qquad (e\,2.24)$$

$$\approx_{\delta+\eta/4} (1-p)c(1-p) + pcp = c \approx_\delta V^*U^*cUV \qquad (e\,2.25)$$

$$\text{for all } c \in \mathcal{G}_1 \text{ and} \qquad (e\,2.26)$$

$$W^*V^*U^*bUVW = p(W')^*V^*U^*bUVW'p = pbp = b \text{ for all } b \in B_2. \qquad (e\,2.27)$$

In particular, (by applying (e 2.18)),

$$pU'VW = U'VWp \text{ and } (pU'VWp)b = bpU'VWp \text{ for all } b \in B_2. \tag{e 2.28}$$

Let

$$W_{00} = (pU'VWp)^* \in U(pAp). \tag{e 2.29}$$

By replacing W by $W((1-p) \oplus W_{00})$, we may assume that $pU'VWp = p$. Then, by (e 2.23), we compute that

$$\mathrm{Bott}(U'VW, \varphi) \;\; = \;\; \mathrm{Bott}((1-p)U'VW, \varphi') + \mathrm{Bott}(pU'VW, \varphi'') = 0. \tag{e 2.30}$$

But, by (e 2.17), we also have that

$$\begin{aligned} 0 = \mathrm{Bott}(U'VW, \varphi) \;\; &= \;\; \mathrm{Bott}(U'V, \varphi) + \mathrm{Bott}(W, \varphi) && \text{(e 2.31)} \\ = \mathrm{Bott}(W, \varphi). && && \text{(e 2.32)} \end{aligned}$$

It follows from the choice of δ_1 and $\mathcal{G}_1$ and (e 2.13) that

$$\mathrm{Bott}(W, \mathrm{ad}\, U'V \circ \varphi) = 0. \tag{e 2.33}$$

It follows from 17.5 (see also 8.4) of [24] that there are $Z_1, Z_2, ..., Z_m \in U(A)$ such that

$$\begin{aligned} W = Z_1 Z_2 \cdots Z_m, \;\; \|W_i(V^*U^*bUV) - (V^*U^*bUV)W_i\| < \epsilon/2m \text{ for all } b \in \mathcal{F} && \text{(e 2.34)} \\ \text{and } \|Z_i - 1\| < 2\pi/(m-1), \;\; i = 1, 2, ..., m. && \text{(e 2.35)} \end{aligned}$$

So, by (e 2.5), (note also we assume that $\delta < \epsilon/4m$),

$$\|W_i b - bW_i\| < \epsilon/m, i = 1, 2, ..., m. \tag{e 2.36}$$

Now define

$$\begin{aligned} W_1 &= V_1 Z_1, \;\; X_2 = Z_1^* V_2 Z_1, Z_2, \; ... && \text{(e 2.37)} \\ W_j &= (Z_1 Z_2 \cdots Z_{j-1})^* V_j (Z_1 Z_2 \cdots Z_{j-1}), ..., && \text{(e 2.38)} \\ W_m &= (Z_1 Z_2 \cdots Z_{m-1})^* V_m (Z_1 Z_2 \cdots Z_{m-1}). && \text{(e 2.39)} \end{aligned}$$

We estimate that

$$\begin{aligned} \|W_j - 1\| \;\; &\le \;\; \|(Z_1 Z_2 \cdots Z_{j-1})^* (V_j - 1)(Z_1 Z_2 \cdots Z_{j-1}) Z_l\| + \|Z_j - 1\| && \text{(e 2.40)} \\ &< \;\; \|V_{j-1} - 1\| + \frac{2\pi}{m-1}, \;\; l = 1, 2, ... && \text{(e 2.41)} \end{aligned}$$

Note that

$$W_1 W_2 \cdots X_j = V_1 V_2 \cdots V_j Z_1 Z_2 \cdots Z_j, \;\; j = 1, 2, ..., m \tag{e 2.42}$$

For each $a \in B_2$,

$$(W_1 W_2 \cdots W_m)^* U^* bU (W_1 W_2 \cdots W_m) \;\; = \;\; W^* V^* U^* bUVW = b \text{ for all } b \in B_2. \tag{e 2.43}$$

Moreover, for $c \in \mathcal{F}$,

$$\begin{aligned} &(W_1 W_2 \cdots W_j) c (W_1 W_2 \cdots W_j)^* && \text{(e 2.44)} \\ &= \;\; V_1 V_2 \cdots V_j Z_1 Z_2 \cdots Z_j c (Z_1 Z_2 \cdots Z_j)^* (V_1 V_2 \cdots V_j)^* && \text{(e 2.45)} \\ &\approx_{j\epsilon/m} \;\; (V_1 V_2 \cdots V_j) c (V_1 V_2 \cdots V_j)^*, \;\; j = 1, 2, ..., m. && \text{(e 2.46)} \end{aligned}$$

□

Lemma 2.10. *Let B be a finite dimensional C^*-subalgebra of a unital simple C^*-algebra A. Then for any $\delta > 0$ there exists $\sigma > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: If*

$$\|pf - fp\| < \sigma$$

for all $f \in \mathcal{G}$, then there is an monomorphism $\varphi : B \to pAp$ or $\varphi : B \to 1_B A 1_B$ such that

$$\|pbp - \varphi(b)\| < \delta\|b\| \text{ for all } b \in B.$$

Proof. Write $B = M_{r(1)} \oplus M_{r(2)} \oplus \cdots \oplus M_{r(l)}$. Let $e_{i,j}^{(s)} \in B$ be a system of matrix units for $M_{r(s)}$, $s = 1, 2, ..., l$. Since A is simple, it is easy to obtain, for each i, elements $v_{s,i,1}, v_{s,i,2}, ..., v_{s,i,m(s)}$ in B with $\|v_{s,i,k}\| \le 1$ such that

$$\sum_{k=1}^{m(s)} v_{s,i,k}^* e_{i,i}^{(s)} v_{s,i,k} = 1_A, s = 1, 2, ..., l.$$

Let $\mathcal{F}_0 = \{e_{i,j}^{(s)} : 1 \le s \le l\} \cup \{v_{s,i,k} : 1 \le i \le R(s), 1 \le k \le m(s), 1 \le s \le l,\}$. Let $J = \max\{m(s) : s = 1, 2, ..., l\}$. Let $\eta > 0$ to be determined. Suppose that

$$\|pb - bp\| < (1/2J)^2 \min\{1/4, \delta/4\} \text{ for all } b \in \mathcal{F}_0.$$

Then

$$\|\sum_{k=1}^{m(s)} pv_{s,i,k}^* p e_{i,i}^{(s)} p v_{s,i,k} p - p\| < (1/2J) \min\{1/4, \eta/4\}, s = 1, 2, ..., l.$$

It follows that

$$\|pe_{i,i}^{(s)}p\| > 1/2J, i = 1, 2, ..., R(s), s = 1, 2, ..., l.$$

Put $a = pe_{i,i}^{(s)}p$. We claim that, in fact, $\|a\| \ge 1/2$. Otherwise $\|a\| < 1/2$. We have

$$\|a - a^2\| = \|pe_{i,i}^{(s)}p - (pe_{i,i}^{(s)}pe_{i,i}^{(s)}p)\| < (1/2J)^2 \min\{1/4, \eta/4\}.$$

Applying the spectral theorem, if $t = \|a\|$, we have that

$$|t - t^2| < (1/2J)^2 \min\{1/4, \eta/4\}.$$

It follows that (since $|t| = \|a\| > 1/2J$)

$$|1 - t| < (1/2J)(1/4) \le 1/8.$$

It is impossible unless $t \ge 1/2$.

It then follows from 2.5.5 of [17] that there is a nonzero projection $q_{i,i}^{(s)} \in pAp$ such that

$$\|pe_{i,i}^{(s)}p - q_{1,1}^{(s)}\| < (1/2J)^2 \min\{1/2, \eta/2\}.$$

The rest of proof is standard and follows from the argument in section 2.5 of [17] and 2.3 of [19]. ∎

Lemma 2.11. *Let A be a unital C^*-algebra and let $V, V_1, V_2, ..., V_m \in U_0(A)$ be unitaries such that $V = V_1 V_2 \cdots V_m$.*

Then, for any $\delta > 0$ and any nonzero projection $p \in A$ there is $\eta = \eta(\delta, m) > 0$ (which does not depend on A) satisfying the following: if

$$\|pV - Vp\| < \eta \text{ and } \|pV_i - V_i p\| < \eta, \;\; i = 1, 2, ..., m,$$

then there exist unitaries $W, W_i \in pAp$ such that

$$\|W - pVp\| < \delta, \;\; \|W_i - pV_i p\| < \delta \text{ and } W = W_1 W_2 \cdots W_m.$$

The proof of the above is standard (see for example section 2.5 of [17]).

Lemma 2.12. *Let A be a unital C^*-algebra. Let $k > 1$ be an integer. Suppose that there are k mutually orthogonal projections $e_1, e_2, ..., e_k$ in A and a unitary $u \in U(A)$ such that*

$$u^* e_i u = e_{i+1}, \;\; i = 1, 2, ..., k \quad \text{and} \;\; e_{k+1} = e_1.$$

Suppose that B is a finite dimensional C^*-subalgebra in $e_1 A e_1$ and $z \in U_0(e_1 A e_1)$ such that $z^*(u^k)^* b u^k z = b$ for all $b \in B$. Suppose that $z = z_1 z_2 \cdots z_{k-1}$, where $z_i \in U(e_1 A e_1)$, $i = 1, 2, ..., k-1$ nd $z_k = 1$. Define

$$w = \sum_{i=1}^{k} e_i u^{k+1-i} z_i (u^{k-i})^* + (1 - \sum_{i=1}^{k} e_i) u.$$

Then

$$\begin{aligned} &\|w - u\| \le \max\{\|z_i - 1\| : 1 \le i \le k-1\} \\ &(w^i)^* e_1 (w^i) = e_{i+1}, \quad i = 1, 2, ..., k-1 \;\; \text{and} \\ &(w^k)^* b w^k = b \qquad \text{for all} \;\; b \in B. \end{aligned} \tag{e 2.47}$$

Proof. It is easy to verify that w is a unitary. One estimates that

$$\|w - u\| \le \max\{\|z_i - 1\| : 1 \le i \le k-1\}.$$

Note that

$$\begin{aligned} (w^i)^* e_1 w^i &= u^{k-i} (z_1 z_2 \cdots z_i)^* (u^k)^* e_1 u^k (z_1 z_2 \cdots z_i)(u^{k-i})^* \\ &= u^{k-i} e_1 (u^{k-i})^* = e_{i+1}, \qquad\qquad i = 1, 2, ..., k. \end{aligned}$$

Moreover,

$$\begin{aligned} e_1 w^k &= e_1 u^k (z_1 z_2 \cdots z_{k-1}) \\ &= e_1 u^k z. \end{aligned} \tag{e 2.48}$$

Then

$$(w^k)^* b w^k = z^* (u^k)^* b u^k z = b \;\; \text{for all} \;\; b \in B.$$

The lemma then follows.

□

Lemma 2.13. *Let A be a unital C^*-algebra. Suppose that $e_1, e_2, ..., e_n$ are n mutually orthogonal projections and $u \in U(A)$ is a unitary such that*

$$u^* e_i u = e_{i+1}, \;\; i = 1, 2, ..., n \;\; \text{and} \;\; e_{n+1} = e_1.$$

Then $\{e_i : i = 1, 2, ..., n\}$, $\{e_i u e_{i+1} : i = 1, 2, ..., n-1\}$, $\{pup\}$, where $p = \sum_{i=1}^n e_i$, generate a C^*-subalgebra which is isomorphic to $C(X) \otimes M_n$, where X is a compact subset of S^1.

Proof. It is standard to check that $\{e_i : i = 1, 2, ..., n\}$ and $\{e_i u e_{i+1} : i = 1, 2, ..., n-1\}$ generate a C^*-subalgebra which is isomorphic to M_n. Let $z = e_1 u^n e_1$. Since $e_1 u^n = u^n e_1$, z is a unitary in $e_1 A e_1$. Suppose that $sp(z) = X \subset S^1$. Then $\{e_i : i = 1, 2, ..., n\}$, $\{e_i u e_{i+1} : i = 1, 2, ..., n-1\}$ and z generate a C^*-subalgebra B which is isomorphic to $C(X) \otimes M_n$. Note

$$e_1 u^{n-1} e_n = e_1 u e_2 u e_3 \cdots e_{n-1} u e_n \in B.$$

We also have that

$$e_n u e_1 = (e_1 u^{n-1} e_n)^* z \in B.$$

So

$$pup = \sum_{i=1}^{n-1} e_i u e_{i+1} + e_n u e_1 \in B.$$

On the other hand, the C^*-subalgebra generated by $\{e_i : i = 1, 2, ..., n\}$, $\{e_i u e_{i+1} : i = 1, 2, ..., n-1\}$ and $\{pup\}$ contains $e_n u e_1$ as well as z. This proves the lemma. $\square$

3 Tracial rank zero

Lemma 3.1. *Let A be a unital separable simple C^*-algebra with tracial rank zero and let $G \subset K_0(A)$ be a subgroup such that $\rho_A(G)$ is dense in $\rho_A(K_0(A))$. Then, for any $\epsilon > 0$ any finite subset $\mathcal{F} \subset A$ and any nonzero positive element $b \in A_+$, there exists a projection $p \in A$ and a finite dimensional C^*-subalgebra $B \subset A$ with $1_B = p$ and $[e] \in G$ for all projections $e \in B$ such that*

(1) $\|pa - ap\| < \epsilon$ for all $a \in \mathcal{F}$,
(2) *$pap \in_\epsilon B$ for all $a \in \mathcal{F}$ and*
(3) $[1-p] \le [b]$.

Proof. Fix any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A$ and any nonzero positive element $b \in A_+$. There are mutually orthogonal nonzero projections $r_1, r_2 \in \overline{bAb}$. Since $TR(A) = 0$, there is a projection $q \in A$ and there is a finite dimensional C^*-subalgebra $B_1 \subset A$ with $1_{B_1} = q$ such that

(1) $\|qa - aq\| < \epsilon$ for all $a \in \mathcal{F}$,
(2) $qaq \in_\epsilon B_1$ for all $a \in \mathcal{F}$ and
(3) $[1-q] \le [r_1]$.

Write $B_1 = M_{R(1)} \oplus M_{R(2)} \oplus \cdots \oplus M_{R(k)}$. Let $\{e_{i,j}^{(l)}\}$ be a system of matrix units, $l = 1, 2, ..., k$. Put $\delta = \inf\{\tau(r_2) : \tau \in T(A)\}$. Note that $\delta > 0$. Since $TR(A) = 0$ and $\rho_A(G)$ is dense in $\rho_A(K_0(A))$, there is, for each l, a nonzero projection $d_l \le e_{1,1}^{(l)}$ such that $[d_l] \in G$ and

$$\tau(e_{1,1}^{(l)} - d_l) < \frac{\delta \cdot \inf\{\tau(e_{1,1}^{(l)} : \tau \in T(A)\}}{R(l)} \text{ for all } \tau \in T(A), \ \ l = 1, 2, ..., k.$$

Define

$$p_l = \sum_{i=1}^{R(l)} e_{i,1}^{(l)} d_l e_{1,l}^{(l)}, \ \ l = 1, 2, ..., k.$$

In other words, p_l has the form $\mathrm{diag}(\overbrace{d_l, d_l, \cdots, d_l}^{R(l)})$. It follows that $[p_l] \in G$ and p_l commutes with every element in B_1. We compute that

$$\tau(\sum_{i=1}^{R(l)} e_{i,i}^{(l)} - p_l) < \delta \cdot \inf\{\tau(e_{1,1}^{(l)} : \tau \in T(A)\} \text{ for all } \tau \in T(A), \ \ l = 1, 2, ..., k.$$

Define

$$p = \sum_{l=1}^{k} p_l \text{ and } B = pB_1p.$$

It follows that

$$\tau(q-p) < \delta < \tau(r_2) \text{ for all } \tau \in T(A).$$

Therefore, by [15], $[p-q] \leq [r_2]$. On the other hand, since $[d_l] \in G$, we see that $[e] \in G$ for all projections $e \in B$.

Moreover, we have the following:
(i) $\|pa - ap\| < \epsilon$ for all $a \in \mathcal{F}$,
(ii) $pap \in_\epsilon B$ for all $a \in \mathcal{F}$ and
(iii) $[1-p] \leq [r_1] + [r_2] \leq [b]$.

□

Lemma 3.2. *Let A be a unital separable simple C^*-algebra with real rank zero. Suppose that A has the following property:*

For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any nonzero positive element $b \in A_+$, there exists a projection $p \in A$ and a C^*-subalgebra $D \subset A$ with $1_D = p$ and $D \cong \oplus_{j=1}^N C(X_j) \otimes F_j$, where each F_j is a finite dimensional C^*-subalgebra and $X_j \subset S^1$ is a compact subset, such that
(1) $\|pa - ap\| < \epsilon$ *for all* $a \in \mathcal{F}$,
(2) $pap \in_\epsilon D$ *for all* $a \in \mathcal{F}$ *and*
(3) $[1-p] \leq [b]$.

Then A has tracial rank zero.

Proof. This is known. We sketch the proof as follows: Since A has real rank zero, $C(X_j) \otimes F_j$ is approximated pointwise in norm by a finite dimensional C^*-subalgebra, if $X_j \neq S^1$. It is then easy to see , with (1), (2) and (3) above, that $TR(A) \leq 1$ (by replacing ϵ by 2ϵ and replacing $C(X_j) \otimes F_j$ by a finite dimensional C^*-subalgebra, if $X_j \neq S^1$) (see also 6.13 of [15]).

It follows from (b) of Theorem 7.1 of [15] that A is TAI (see the definition of TAI, for example, right above 7.1 of [15]). Since A has real rank zero, any C^*-subalgebra with the form $M_m(C(I))$, where $I = [0,1]$, is approximated by finite dimensional C^*-subalgebras. It follows that $TR(A) = 0$. □

Remark 3.3. An alternative proof (without using Theorem 7.1 of [15]) is described below:

First show that A has stable rank one (the same proof as that $TR(A) \leq 1$ implies that A has stable rank one). Or, as above, first show that $TR(A) \leq 1$. Then one concludes that A has stable rank one.

Let $u \in A$ be a unitary such that $sp(u) = S^1$. So we have a monomorphism $h_1 : C(S^1) \to A$. Let $\mathcal{G} \subset C(S^1)$ be any finite subset and $\epsilon > 0$. Let $\delta > 0$ be also given. For any $\eta > 0$ and for any projection $r \in A$, there is a projection e and a unitary $v \in (1-e)A(1-e)$ such that $[e] \leq [r]$ and

$$\|u - (e+v)\| < \eta.$$

Write $e = e_1 + e_2$, where e_1, e_2 are two non-zero mutually orthogonal projections. Since A has stable rank one, choose $v_1 \in e_1Ae_1$ and $v_2 \in e_2Ae_2$ such that $[v_2] = [v^*] = [u^*]$ and $[v_1] = [v] = [u]$ in $K_1(A)$. Put $w = v_1 + v_2 + v$. Verify that $[u] = [w]$ in $K_1(A)$ and verify that

$$|\tau(f(u)) - \tau(f(w))| < \delta \text{ for all } \tau \in T(A) \text{ and } \text{ for all } f \in \mathcal{G}$$

if η is sufficiently small and $\sup\{\tau(r) : \tau \in T(A)\}$ is sufficiently small. It follows from, for example, Theorem 3.3 of [22], that there is a unitary $z \in A$ such that

$$\|u - z^*wz\| < \epsilon$$

if δ is sufficiently small.

In other words, there is a unitary $U \in A$ with the form $U = U_1 + U_2$, where $U_1 \in qAq$ and $U_2 \in (1-q)A(1-q)$ are unitaries for which $[q] \le [e_1] \le [r]$ and $[U_2] = 0$ in $K_1(A)$, such that

$$\|u - U\| < \epsilon.$$

Since A is assumed to have real rank zero, $(1-q)A(1-q)$ has real rank zero. It follows from [14] that U_2 can be approximated in norm by unitaries with finite spectrum. From this and together with (1),(2) and (3) above, one easily sees that $TR(A) = 0$.

Theorem 3.4. *Let A be a unital separable amenable simple C^*-algebra with $TR(A) = 0$ which satisfies the UCT. Suppose that $\alpha \in Aut(A)$ has the tracial cyclic Rokhlin property. Suppose also that there is an integer $J \ge 1$ such that $[\alpha^J] = [\mathrm{id}_A]$ in $KL(A, A)$. Then $TR(A \rtimes_\alpha \mathbb{Z}) = 0$.*

Proof. We first note that, by [10], $A \rtimes_\alpha \mathbb{Z}$ is a unital simple C^*-algebra. By [33], $A \rtimes_\alpha \mathbb{Z}$ has real rank zero. We will show that $(A \rtimes_\alpha \mathbb{Z}) \otimes Q$ has tracial rank one, where Q is the UHF-algebra with $(K_0(Q), [1_Q]) = (\mathbb{Q}, 1)$. Let $u \in A \rtimes_\alpha \mathbb{Z}$ be a unitary which implements α, i.e., $\alpha(a) = u^*au$ for all $a \in A$. Put $B = A \otimes Q$. Note that $(A \rtimes_\alpha \mathbb{Z}) \otimes Q$ is generated by $A \otimes Q$ and elements of the form $u \otimes a$ for $a \in Q$. One can identify 1_A and $1_{A \rtimes \alpha \mathbb{Z}}$. Since $A \otimes Q$ contains elements $1_A \otimes a$ ($a \in Q$), $(A \rtimes_\alpha \mathbb{Z}) \otimes Q$ is also generated by $A \otimes Q$ and $u \otimes 1_Q$. By identifying u with $u \otimes 1_Q$, $(A \rtimes_\alpha \mathbb{Z}) \otimes Q$ is generated by B and u. So, in what follows, we will identify u with $u \otimes 1_Q$. We will first show that $TR((A \rtimes_\alpha \mathbb{Z}) \otimes Q) = 0$.

To this end, let $1 > \epsilon > 0$ and $\mathcal{F} \subset (A \rtimes_\alpha \mathbb{Z}) \otimes Q$ be a finite set. To simplify notation, without loss of generality, we may assume that

$$\mathcal{F} = \mathcal{F}_0 \cup \{u\},$$

where $\mathcal{F}_0 \subset B$ is a finite subset of the unit ball which contains 1_B.

Choose an integer k which is a multiple of J such that $8\pi/(k-2) < \epsilon/256$. Put

$$\mathcal{F}_1 = \mathcal{F}_0 \cup \{u^i a (u^*)^i : a \in \mathcal{F}_0, -k \le i \le k\}. \tag{e 3.49}$$

Fix $b_0 \in ((A \rtimes_\alpha \mathbb{Z}) \otimes Q)_+ \setminus \{0\}$. It follows from Theorem 4 of [8] that $A \rtimes_\alpha \mathbb{Z}$ has property (SP). Thus there is a nonzero projection r_{00} in the hereditary C^*-subalgebra of $(A \rtimes_\alpha \mathbb{Z}) \otimes Q$ generated by b_0.

Let $r_1', r_2' \in B$ be nonzero mutually orthogonal projections. Since $A \rtimes_\alpha \mathbb{Z}$ is simple, it follows from [4] (see also (2) of 3.5.6 and 3.5.7 in [17]) that there are nonzero projections $r_i \le r_i'$, such that $[r_i] \le [r_{00}]$, $i = 1, 2$.

Let $\eta_0 = \frac{\epsilon}{256k^4}$. Since $TR(A) = 0$ and satisfies the UCT, by the classification theorem of [20] and [6], without loss of generality, we may assume that $\mathcal{F}_1 \subset B_1 \subset B$, where $B_1 = \oplus_{i=1}^r C_i$ and C_i has the form described in 2.7. Let $\delta > 0$ and $\mathcal{G} \subset B_1$ be a finite subset in 2.9 corresponding to η_0 (in place of ϵ) and $\mathcal{F}_1$ (in place of $\mathcal{F}$). Since $TR(A) = 0$ and $[\alpha^k] = [\mathrm{ad}_A]$ in $KL(A, A)$,

$$\tau((u^k)^* a u^k) = \tau(a) \text{ for all } a \in B \text{ and } \text{ for all } \tau \in T(B). \tag{e 3.50}$$

(Recall that we identify u with $u \otimes 1_Q$). By the assumption and by applying 3.4 of [22] (see also 3.6 of [23]), there is a unitary $w \in U(B)$ such that

$$w^*(u^k)^* 1_{C_j} u^k w = 1_{C_j}, \;\; j = 1, 2, ..., r \text{ and } \tag{e 3.51}$$

$$w^*(u^k)^* c u^k w \approx_{\delta/2} c \text{ for all } c \in \mathcal{G}. \tag{e 3.52}$$

Since $K_1(B)$ is divisible, $H_1(K_0(B_1), K_1(B)) = K_1(B)$. From 2.5, we may assume that $w \in U_0(B)$. There are unitaries $w_1, w_2, ..., w_{k-1} \in U(B)$ such that

$$\|w_i - 1\| < \pi/(k-1) \text{ and } w = w_1 w_2 \cdots w_k. \tag{e 3.53}$$

Define

$$\mathcal{F}_2 = \{u^{-i}bu^i : b \in \mathcal{F}_1 \cup \mathcal{G}, -k \le i \le k\} \text{ and}$$

$$\mathcal{F}_3 = \{u^i(w_{i_1}w_{i_1+1}\cdots w_{i_1+l})^{i_2}a((w_{i_1}w_{i_1+1}\cdots w_{i_1+l})^{i_2})^*u^i : \tag{e 3.54}$$
$$b \in \mathcal{F}_2 \cup \mathcal{G}, -k \le i, i_1, i_2 \le k\} \cup \{w, w_j : 1 \le j \le k\}. \tag{e 3.55}$$

Let $\eta_1 = \min\{\eta_0/4k, \delta/4k\}$.

We will again apply the classification theorem ([20] and [6]). In particular, A is an inductive limit of C^*-algebras in 2.7 with injective connecting maps and with real rank zero. We may assume that there is a projection $p_1 \in B$ such that $p_1x = xp_1$ for all $x \in B_1$ and a finite dimensional C^*-subalgebra B_2 with $1_{B_2} = p_1$ such that $p_1x \in B_2$ for all $x \in B_1$ and $x \mapsto (1-p_1)x(1-p_1)$ is injective on B_1, and

(a) $\|p_1f - fp_1\| < \frac{\eta_1}{64k}$ for all $f \in \mathcal{F}_3$,

(b) $p_1fp_1 \in_{\frac{\eta_1}{64k}} B_2$ for all $f \in \mathcal{F}_3$ and

(c) $[1-p_1] \le [r_1]$.

Let $\varphi' : B_1 \to (1-p_1)A(1-p_1)$. Put $B_3 = B_2 \oplus \varphi'(B_1)$. It follows from 2.9 that there exist unitaries $W, W_1, W_2, ..., W_{k-1} \in B$ such that

$$W^*(u^k)^*au^kW = a \text{ for all } a \in B_2, \tag{e 3.56}$$
$$W = W_1W_2\cdots W_{k-1}, \ \|W_i - 1\| < 2\pi/(k-2) + \pi/(k-2) \text{ and} \tag{e 3.57}$$
$$(W_1W_2\cdots W_l)b(W_1W_2\cdots W_l)^* \approx_{\eta_0} (w_1w_2\cdots w_l)b(w_1w_2\cdots w_l)^* \tag{e 3.58}$$

for all $b \in \mathcal{F}_1$. Set

$$\mathcal{F}_4 = \mathcal{F}_3 \cup \{u^i(W_1W_2\cdots W_{i_1})^*u^ja(u^i(W_1W_2\cdots W_{i_1})^*u^j)^* : a \in \mathcal{F}_4 : -k \le i, i_1, j \le k\}.$$

Let $\sigma_1 > 0$ and $\mathcal{G}_1$ be associated with B_2 and $\eta_1/2$ in 2.10. Let $\eta_2 = \eta(\eta_1/2, k)$ be as in 2.11. Let $\eta_3 = \min\{\eta_2, \sigma_1, \eta_0\}$. Define $\mathcal{F}_5 = \mathcal{F}_4 \cup \mathcal{G}_1$.

Since α has the tracial cyclic Rokhlin property, there exist projections $e_1, e_2, ..., e_k \in B$ such that

(1) $\|\alpha(e_i) - e_{i+1}\| < \frac{\eta_3}{64k}$ for $1 \le i \le k$ ($e_{k+1} = e_1$)

(2) $\|e_ia - ae_i\| < \frac{\eta_3}{64k}$ for $a \in \mathcal{F}_5$

(3) $[1 - \sum_{i=1}^k e_i] \le [r_2]$

(e_i has the form $e_i \otimes 1_Q$).

Set $p = \sum_{i=1}^k e_i$. From (1) above, one estimates that

$$\begin{aligned}\|up - pu\| &= \|\textstyle\sum_{i=1}^k ue_{i+1} - \sum_{i=1}^k e_iu\| \\ &= \textstyle\sum_{i=1}^k \|ue_{i+1} - e_iu\| = \sum_{i=1}^k \|ue_{i+1} - uu^*e_iu\| < \frac{\eta_3}{64}\end{aligned}$$

By (1) above, one sees that there is a unitary $v \in B$ such that

$$\|v - 1\| < \frac{\eta_3}{32k} \text{ and } vu^*e_iuv^* = e_{i+1}, \ \ i = 1, 2, ..., k. \tag{e 3.59}$$

Set $u_1 = v^*u$. Then

$$u_1^*e_iu_1 = e_{i+1}, \ \ i = 1, 2, ...k \text{ and } e_{k+1} = e_1.$$

In particular,

$$u_1^ke_1 = e_1u_1^k.$$

For any $a \in \mathcal{F}_5 \cap B_2$ (since $W \in \mathcal{F}_5$),

$$e_1 W^* e_1 (u_1^k)^* e_1 a e_1 u_1^k e_1 W e_1 \approx_{\eta_3/16k} e_1 a e_1.$$

By 2.10, there is a monomorphism $\varphi_1 : B_2 \to e_1 B e_1$ such that

$$\|\varphi_1(a) - e_1 a e_1\| < \frac{\eta_1}{2}\|a\| \text{ for all } a \in B_2. \tag{e 3.60}$$

By applying 2.11, and using (e 3.60), we obtain unitaries $x, x_1, x_2, ..., x_{k-1} \in U_0(e_1 B e_1)$ such that

$$\|x - e_1 W e_1\| < \eta_1/2, \quad \|x_i - e_1 W_i e_1\| < \eta_1/2 \tag{e 3.61}$$

$$x = x_1 x_2 \cdots x_{k-1} \text{ and } x^*(u_1)^* a u_1 x = a$$

for all $a \in \varphi_1(B_2)$.

Let $Z = \sum_{i=1}^k e_i u_1^{k+1-i} x_i (u_1^{k-i})^* + (1-p)u_1$. Define $B_4 = \varphi_1(B_2)$. As in 2.12, by (e 3.57),

$$\|Z - u_1\| < \eta_1/2 + 3\pi/(k-2), \ (Z^k)^* b Z^k = b \text{ for all } b \in B_4 \tag{e 3.62}$$

and $(Z^i)^* e_1 Z^i = e_{i+1}$, $i = 1, 2, ..., k$ ($e_{k+1} = e_1$).

Write $B_4 = F_1 \oplus F_2 \oplus \cdots \oplus F_N$ and let $\{c_{is}^{(j)}\}$ be the matrix units for F_j, $j = 1, 2, ..., N$, where $F_j = M_{R(j)}$, and put $q = 1_{B_4}$.

Define $D_0 = B_4 \oplus \oplus_{i=1}^{k-1} Z^{i*} B_4 Z^i$ and D_1 the C^*-subalgebra generated by B_4 and $c_{ss}^{(j)} Z^i$, $s = 1, 2, ..., R(j)$, $j = 1, 2, ..., N$ and $i = 0, 1, 2, ..., k-1$. Then $D_1 \cong B_4 \otimes M_k$ and $D_1 \supset D_0$.

Define $q_{ss}^{(j)} = \sum_{i=0}^{k-1} Z^{i*} c_{ss}^{(j)} Z^i$, $q^{(j)} = \sum_{s=1}^{R(j)} q_{ss}^{(j)}$ and $Q = \sum_{j=1}^N q^{(j)} = 1_{D_1}$. Note that $Q = \sum_{i=0}^{k-1} (Z^i)^* q Z^i$. Note that

$$q_{ss}^{(j)} Z = (\sum_{i=0}^{k-1} Z^{i*} c_{ss}^{(j)} Z^i) Z = Z \sum_{i=0}^{k} (Z^{i+1})^* c_{ss}^{(j)} Z^{i+1} \tag{e 3.63}$$

$$= Z(\sum_{i=1}^{k-1} Z^{i*} c_{ss}^{(j)} Z^i + c_{ss}^{(j)}) = Z q_{ss}^{(j)}. \tag{e 3.64}$$

It follows from 2.13 that $c_{11}^{(j)}$, $c_{11}^{(j)} Z^i$ and $c_{11}^{(j)} Z^k c_{11}^{(j)}$ generate a C^*-subalgebra which is isomorphic to $C(X_j) \otimes M_k$ for some compact subset $X_j \subset S^1$. Moreover, $q_{ss}^{(j)} Z q_{ss}^{(j)}$ is in the C^*-subalgebra. Let D be the C^*-subalgebra generated by D_1 and $c_{11}^{(j)} Z^k c_{11}^{(j)}$. Then $D \cong \oplus_{j=1}^N C(X_j) \otimes B_4 \otimes M_k$. It follows from (e 3.63) that $q^{(j)}$ and Q commutes with Z. Therefore $QZQ \in D$. Thus, by (e 3.62),

$$\|Qu - uQ\| \le \|Qu - Qu_1\| + \|Qu_1 - QZ\| + \|ZQ - u_1 Q\| + \|u_1 Q - uQ\| \tag{e 3.65}$$

$$< 2(\eta_3/32k + (3\pi/(k-2) + \eta_1/2)) < \epsilon. \tag{e 3.66}$$

From $QZQ \in D$, we also have

$$QuQ \in_\epsilon D. \tag{e 3.67}$$

For $b \in \mathcal{F}_0$, by (e 3.61) and (e 3.59), we estimate that

$$\begin{aligned} & (Z^i)^* q (Z^i) b \approx_{2k\eta_1/2 + k\eta_3/32k} (Z^i)^* q u_1^k (W_1 W_2 \cdots W_i)(u^{k-i})^* b \\ = \quad & (Z^i)^* q u_1^k (W_1 W_2 \cdots W_i)(u^{k-i})^* b u^{k-i} (W_1 W_2 \cdots W_i)^* (u_1^k)^* [u^{k-i} (W_1 W_2 \cdots W_i)^* (u_1^k)^*]^* \end{aligned}$$

Put $c_i = (u^{k-i})^* b u^{k-i}$. Then $c_i \in \mathcal{F}_1$. Note that we have assumed that $\mathcal{F}_1 \subset B_1$. In particular, $p_1 c_i = c_i p_1$.

Since $(w_1 w_2 \cdots w_i)\mathcal{F}_1 (w_1 w_2 \cdots w_i)^* \subset \mathcal{F}_3$, by (e 3.58),

$$\begin{aligned}
&(Z^i)^* q u_1^k (W_1 W_2 \cdots W_i)(u^{k-i})^* b u^{k-i} (W_1 W_2 \cdots W_i)^* (u_1^k)^* [u^{k-i}(W_1 W_2 \cdots W_i)^* (u_1^k)^*]^* \\
&= (Z^i)^* q u_1^k (W_1 W_2 \cdots W_i) c_i (W_1 W_2 \cdots W_i)^* (u_1^k)^* [u^{k-i}(W_1 W_2 \cdots W_i)^* (u_1^k)^*]^* \\
&\approx_{\eta_0} (Z^i)^* q u_1^k (w_1 w_2 \cdots w_i) c_i (w_1 w_2 \cdots w_i)^* u_1^k [u^{k-i}(W_1 W_2 \cdots W_i)^* (u_1^k)^*]^* \\
&\approx_{2k\eta_3/32k+\eta_1/32k+\eta_3/32k} (Z^i)^* u_1^k (w_1 w_2 \cdots w_i) c_i (w_1 w_2 \cdots w_i)^* u_1^k q [u^{k-i}(W_1 W_2 \cdots W_i)^* u_1^k]^* \\
&\approx_{\eta_0} (Z^i)^* u_1^k (W_1 W_2 \cdots W_i) c_i (W_1 W_2 \cdots W_i)^* (u_1^k)^* q [u^{k-i}(W_1 W_2 \cdots W_i)^* (u_1^k)^*]^* \\
&= (Z^i)^* u_1^k (W_1 W_2 \cdots W_i)(u^{k-i})^* b u^{k-i} (W_1 W_2 \cdots W_i)^* (u_1^k)^* q [u^{k-i}(W_1 W_2 \cdots W_i)^* (u_1^k)^*]^* \\
&\approx_{2k\eta_3/32k} (Z^i)^* u_1^k (W_1 W_2 \cdots W_i)(u_1^{k-i})^* b u_1^{k-i} (W_1 W_2 \cdots W_i)^* (u_1^k)^* q [u^{k-i}(W_1 W_2 \cdots W_i)^* (u_1^k)^*]^* \\
&\approx_{2k\eta_1/2} b (Z^i)^* q Z^i.
\end{aligned}$$

Note that ($k \geq 2 + (4 \cdot 256)\pi/\epsilon$)

$$\begin{aligned}
(2k\eta_1/2 + \eta_3/32) + \eta_0 + \eta_1/32 + \eta_1/32k + \eta_3/32k \quad &\text{(e 3.68)} \\
+\eta_0 + \eta_3/16 + 2k\eta_1/2 < \epsilon/k^3. \quad &\text{(e 3.69)}
\end{aligned}$$

Hence

$$\|(Z^i)^* q Z^i b - b (Z^i)^* q Z^i\| < \epsilon/k^3, \ \ k = 0, 1, ..., k-1. \tag{e 3.70}$$

Therefore, for $b \in \mathcal{F}_0$,

$$\|Qb - bQ\| < k(\epsilon/k^3) = \epsilon/k^2. \tag{e 3.71}$$

It follows from (e 3.67) and (e 3.66) that

$$\|Qa - aQ\| < \epsilon \quad \text{for all} \ \ a \in \mathcal{F}. \tag{e 3.72}$$

For any $b \in \mathcal{F}_0$, a similar estimation above shows that

$$\begin{aligned}
&\|q Z^i b (Z^i)^* q - q u_1^k (w_1 w_2 \cdots w_i)(u^{k-i})^* b u^{k-i} (w_1 w_2 \cdots w_i)^* (u_1^k)^* q\| \\
&\qquad < 2(2k\eta_1/2 + k\eta_3/32k) = 2k\eta_1 + \eta_3/16.
\end{aligned}$$

However, by (e 3.59), (e 3.60), (b) and (e 3.60),

$$q u_1^k (w_1 w_2 \cdots w_i)(u^{k-i})^* b u^{k-i} (w_1 w_2 \cdots w_i)^* (u_1^k)^* q \in_{2\eta_3/32k+\eta_1/2+\eta_1/64+\eta_1/2} B_4.$$

It follows that, for $b \in \mathcal{F}_0$,

$$(Z^i)^* q Z^i b (Z^i)^* q Z^i \in_{\epsilon/k^3} (Z^i)^* B_4 Z^i, \ \ i = 0, 1, 2, ..., k-1. \tag{e 3.73}$$

Combing (e 3.73) with (e 3.70), we obtain that, for $b \in \mathcal{F}_0$,

$$QbQ \in_{k^2\epsilon/k^3} D_1 \subset D. \tag{e 3.74}$$

Combing with (e 3.67), we obtain that

$$QaQ \in_\epsilon D \ \text{ for all } \ a \in \mathcal{F}. \tag{e 3.75}$$

We also compute that

$$
\begin{aligned}
[1-Q] &\leq [1-\sum_{i=1}^{k} e_i]+[1-p_1] \\
&\leq [r_2]+[r_1]\leq [r_{00}]\leq [b_0]. \qquad \text{(e 3.76)}
\end{aligned}
$$

Combing (e 3.72), (e 3.75) and (e 3.76), by 3.2, we conclude that $TR((A\rtimes_\alpha\mathbb{Z})\otimes Q)=0$. It follows from Theorem 3.6 of [29] that $TR((A\rtimes_\alpha\mathbb{Z})\otimes M)=0$ for any UHF-algebra M of infinite type. On the other hand, by [33], $A\rtimes_\alpha\mathbb{Z}$ has real real rank zero, stable rank one and weakly unperforated $K_0(A\rtimes_\alpha\mathbb{Z})$. It follows from a classification result (Theorem 5.4 of [28]) that $(A\rtimes_\alpha\mathbb{Z})\otimes\mathcal{Z}$ is isomorphic to a unital simple C^*-algebra with tracial rank zero. Since $\tau\circ\alpha^J=\tau$ for all $\tau\in T(A)$, the tracial cyclic Rokhlin property clearly implies the weak Rokhlin property. By 4.9 of [31], $A\rtimes_\alpha\mathbb{Z}$ is $\mathcal{Z}$ stable. It follows that $TR(A\rtimes_\alpha\mathbb{Z})=0$.

□

Remark 3.5. The proof presented above corrects 2.9 of [27]. If $\alpha^k=\mathrm{id}_A$ for some $k>1$, however, the proof of 2.9 of [27] works by replacing 2 by k (with $D\cong e_1Ae_1\otimes M_k$). Since in this case the claim that $pwp\in D$ remains valid, conclusion of 2.9 of [27] holds. One should also note that if α^J is approximately inner for some integer $J>0$, then $[\alpha^J]=[\mathrm{id}_A]$ in $KL(A,A)$ An early version of the proof of Theorem 3.4 contains an error which was pointed us by Hiroki Matui. So it is appropriate to acknowledge this at this point and to choose their recent result ([31]) to simplify the proof. More general result related to 3.4 will be discussed elsewhere.

Proposition 3.6. *Let A be a unital separable simple C^*-algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$ with unique tracial state. Let $\alpha\in Aut(A)$ have the tracial cyclic Rokhlin property. Then there exists a subgroup $G\subset K_0(A)$ such that $\alpha_{*0}^2(g)=g$ for all $g\in G$ and $\rho_A(G)$ is dense in $Aff(K_0(A))$.*

Proof. Let $G\subset K_0(A)$ such that $\alpha_{*0}^2(g)=g$ for all $g\in G$. Then $[1_A]\in G$. Let $\tau\in T(A)$ be the unique tracial state. To show that $\rho_A(G)$ is dense in $\rho_A(K_0(A))$, it suffices to show that, for any $n>0$, there exists a nonzero projection $e\in A$ such that $[\alpha^2(e)]=[e]$ and $\tau(e)<1/n$.

Therefore it suffices to show the following, for any $1>\epsilon>0$ and any $0<\delta<\frac{1-\epsilon}{4}$, if $p\in A$ is a nonzero projection such that

$$\|\alpha^2(p)-p\|<\epsilon$$

there is a nonzero projection $q\leq p$ such that $p-q\neq 0$,

$$\|\alpha^2(q)-q\|<\epsilon+\delta \text{ and } \|\alpha^2(p-q)-(p-q)\|<\epsilon+\delta.$$

Because we must have that $[p],[q],[p-q]\in G$, that either $\tau(p-q)\leq\frac{\tau(p)}{2}$ or $\tau(q)\leq\frac{\tau(p)}{2}$ and that $\epsilon+\delta<1$.

There is at least one such p (namely, 1_A).

For any finite subset $(p\in)\mathcal{G}\subset A$ and $0<\eta<\min\{\frac{\delta}{256},\frac{1-\epsilon}{256},\frac{\epsilon}{256}\}$, there are nonzero mutually orthogonal projections $f_i, i=1,...,4$ such that

(i) $\|\alpha(f_i)-f_{i+1}\|<\eta$, $i=1,2,...,4$ with $f_5=f_1$,

(ii) $\|f_ia-af_i\|<\eta$ for all $a\in\mathcal{G}$ and

(iii) $\tau(1-\sum_{i=1}^4 f_i)<\eta$.

Put $e=f_1+f_3$. Then

$$\|\alpha^2(e)-e\|<2\eta.$$

By taking sufficiently large $\mathcal{G}$ and sufficiently small η, by applyng 2.10, there exist nonzero projections $q,q_1\leq p$ such that

$$\|q-epe\|<2\eta \text{ and } \|q_1-(1-e)p(1-e)\|<2\eta$$

We compute that

$$\begin{aligned}\|\alpha^2(q)-q\| &\le \|\alpha^2(q)-\alpha^2(epe)\|+\|\alpha^2(epe)-epe\|\\ &\le 2\eta+\|\alpha^2(e)\alpha^2(p)\alpha^2(e)-epe\|<2\eta+2\eta+\eta+\epsilon<\epsilon+\delta/51<\epsilon+\delta\end{aligned}$$

A similarly argument also shows that

$$\|\alpha^2(p-q)-(p-q)\|<\epsilon+\delta.$$

□

4 Tracial Rokhlin Property

Let A be a unital separable simple C^*-algebra with $TR(A)=0$ and let α be an automorphism on A. We now turn to the question when α has the tracial cyclic Rokhlin property.

Let X be a compact metric space, let $\sigma: X\to X$ be a homeomorphism and let μ be a normalized σ-invariant Borel measure. Recall that σ has the Rokhlin property if, for any $\epsilon>0$ and any $n\in\mathbb{N}$, there exists a Borel set $E\subset X$ such that $E,\sigma(E),...,\sigma^n(E)$ are mutually disjoint and

$$\mu(X\setminus\cup_{i=0}^n\sigma^i(E))<\epsilon.$$

From this, one may argue that the tracial Rokhlin property for automorphism α above is natural generalization of the commutative case. In fact, from Theorem 2.3, it seems that the tracial Rokhlin property occurs more often than one may first thought and it appears that it is a rather natural phenomenon in the context of automorphisms on simple C^*-algebras.

Kishimoto originally studied approximately inner (but outer) automorphisms regarding Problem **P1**. It was recently proved by N. C. Phillips that for a unital separable simple C^*-algebra with tracial rank zero there is a dense G_δ-set of approximately inner automorphisms which satisfy the tracial Rokhlin property. This shows that automorphisms with the tracial Rokhlin property are abundant. But do they also have the tracial cyclic Rokhlin property? It is proved (also follows from Theorem 4.5 below) that if α is approximately inner as in the Kishimoto's original case, tracial Rokhlin property implies the tracial cyclic Rokhlin property.

Lemma 4.1. *Let A be a unital separable simple C^*-algebra with real rank zero and stable rank one and let $\alpha\in Aut(A)$. Suppose that there is a subgroup $G\subset K_0(A)$ such that $\rho_A(G)$ is dense in $\rho_A(K_0(A))$ and $(\alpha)_{*0}|_G=\mathrm{id}_G$. Then*

$$\tau(a)=\tau(\alpha(a))$$

for all $a\in A$.

Proof. Let $p\in A$ be a projection. Since $\rho_A(G)$ is dense in $\rho_A(K_0(A))$ and A is simple and has real rank zero and stable rank one, there are projections $p_n\le p$ such that

$$[p_n]\in G \text{ and } \lim_{n\to\infty}\sup\{\tau(p-p_n):\tau\in T(A)\}=0.$$

Similarly, there are projections $e_n\le 1-(p-p_n)$ such that

$$[e_n]\in G \text{ and } \lim_{n\to\infty}\sup\{\tau(1-(p-p_n)-e_n):\tau\in T(A)\}=0.$$

It follows that

$$\lim_{n\to\infty}\sup\{\tau(1-e_n):\tau\in T(A)\}=0.$$

However, $[1-e_n] \in G$, by the assumption,

$$[\alpha(1-e_n)] = [1-e_n] \text{ in } K_0(A).$$

Since A has stable rank one, we have

$$\tau(\alpha(1-e_n)) = \tau(1-e_n) \to 0$$

uniformly on $T(A)$. Since $p - p_n \leq 1 - e_n$, we conclude that

$$\tau(\alpha(p-p_n)) \leq \tau(\alpha(1-e_n)) \to 0$$

uniformly on $T(A)$. Thus, for any $\epsilon > 0$, there exists N such that

$$\tau(p-p_n) < \epsilon/2 \text{ and } \tau(\alpha(p-p_n)) < \epsilon/2$$

for all $n \geq N$ and $\tau \in T(A)$. It follows that

$$\begin{aligned} |\tau(\alpha(p)) - \tau(p)| &\leq |\tau(\alpha(p-p_n)| + |\tau(\alpha(p_n)) - \tau(p_n)| + |\tau(p-p_n)| && \text{(e 4.77)} \\ &< \epsilon/2 + 0 + \epsilon/2 = \epsilon. && \text{(e 4.78)} \end{aligned}$$

Therefore, for any projection $p \in A$,

$$\tau(\alpha(p)) = \tau(p)$$

for all $\tau \in T(A)$.

Since A has real rank zero, the above implies that $\tau(\alpha(a)) = \tau(a)$ for all $a \in A_{s.a}$. The lemma then follows.

□

Theorem 4.2. *Let A be a unital separable amenable simple C^*-algebra with $TR(A) = 0$ which satisfies the UCT and let $\alpha \in Aut(A)$. Suppose that α satisfies the tracial Rokhlin property. If there is an integer $r > 0$ such that $\alpha_{*0}^r|_G = \mathrm{id}_G$ for some subgroup $G \subset K_0(A)$ for which $\rho_A(G)$ is dense in $\rho_A(K_0(A))$, then α satisfies the tracial cyclic Rokhlin property.*

Theorem 4.2 strengthens Theorem 3.14 of [22] slightly. This is done by improving Lemma 6.3 of [22]. The rest of the proof will be exactly the same as that of Theorem 3.14 of [22] (which follows closely an idea of Kishimoto) but applying Lemma 4.4 below.

Lemma 4.3. *Let A be a unital simple separable C^*-algebra with $TR(A) = 0$ and let $\alpha \in Aut(A)$ such that $(\alpha)_{*0}|_G = \mathrm{id}_G$ for some subgroup $G \subset K_0(A)$ for which $\rho_A(G) = \mathrm{Aff}(K_0(A))$. Suppose that $\{p_j\}$ is a central sequence of projections such that $[p_j] \in G$ and define $\varphi_j(a) = p_j a p_j$ and $\psi_n(a) = \alpha(p_j) a \alpha(p_j)$, $j = 1, 2, ...$ Then $\{\varphi_j\}$ and $\{\psi_j\}$ are two sequentially asymptotic morphisms. Suppose also that there are finite-dimensional C^*-subalgebras B_j and $C_j = \alpha(B_j)$ with $1_{B_j} = p_j$ and $1_{C_j} = \alpha(p_j)$ such that $[p_{j,i}] \in G$ for each minimal central projection $p_{j,i}$ of B_j $(1 \leq i \leq k(j))$ and there are sequentially asymptotic morphisms $\{\varphi'_j\}$ and $\{\psi'_j\}$ such that*

$$\begin{aligned} &\varphi'_j(a) \subset B_j, \;\; \psi'_j(a) \subset C_j, \\ &\lim_{n\to\infty} \|\varphi_j(a) - \varphi'_j(a)\| = 0 \text{ and } \lim_{n\to\infty} \|\psi_j(a) - \psi'_j(a)\| = 0 \end{aligned}$$

for all $a \in A$. Then, for any $\epsilon > 0$ and for any finite subset $\mathcal{G} \subset A$ and any finite subste of projections $\mathcal{P}_0 \subset M_k(A)$ (for some $k \geq 1$) for which $[p] \in G$ for all $p \in \mathcal{P}_0$, there exists an integer $J > 0$ such that

$$|\tau \circ \varphi_j(a) - \tau \circ \psi_j(a)| < \epsilon/\tau(p_j) \text{ for all } a \in \mathcal{G}$$

and for all $\tau \in T(A)$, and, for all $j \geq J$,

$$[\varphi_j(p)] = [\psi_j(p)] \text{ in } K_0(A).$$

Proof. The proof is exactly the same as that of Lemma 6.2 of [22]. Instead of applying Lemma 6.1 of [22], we apply 4.1. $\square$

Lemma 4.4. *Let A be a unital separable amenable simple C^*-algebra with $TR(A) = 0$ which satisfies the UCT and let $\alpha \in Aut(A)$ be such that $\alpha_{*0}|_G = \mathrm{id}_G$ for some subgroup G of $K_0(A)$ for which $\rho_A(G) = \rho_A(K_0(A))$. Suppose also that $\{p_j(l)\}$, $l = 0, 1, 2, ..., L$, are central sequences of projections in A such that*

$$p_j(l)p_j(l') = 0 \text{ if } l \neq l' \text{ and } \lim_{j\to\infty} \|p_j(l) - \alpha^l(p_j(0))\| = 0, \ \ 1 \leq l \leq L$$

Then there exist central sequences of projections $\{q_j(l)\}$ and central sequences of partial isometries $\{u_j(l)\}$ such that $q_j(l) \leq p_j(l)$

$$u_j(l)^*u_j(l) = q_j(0), \ \ u_j(l)u_j^*(l) = \alpha^l(q_j(0))$$

for all large j, and

$$\lim_{j\to\infty} \|\alpha^l(q_j(0)) - q_j(l)\| = 0 \text{ and } \lim_{j\to\infty} \tau(p_j(l) - q_j(l)) = 0$$

uniformly on $T(A)$.

Proof. The proof is exactly the same as that of 6.3 of [22]. On page 886 of that proof, it uses the fact that $\rho(G)$ is dense in $\mathrm{Aff}(T(A))$ to produce projection $d_{n(j),t,s}$. This can be done with the current assumption which implies that $\rho(G)$ is dense in $\mathrm{Aff}(T(A))$. The proof of 6.3 of [22] also used Lemma 6.2 of [22]. At the end of p.887, one can apply 4.4 instead of 6.2 of [22]. $\square$

Combining 3.4 with 4.2, we have the following:

Theorem 4.5. *Let A be a unital separable simple amenable C^*-algebra with $TR(A) = 0$ which satisfies the UCT and let α be an automorphism with the tracial Rokhlin property. Suppose also that, for some integer $r > 0$, $\alpha^r_{*0}|_G = \mathrm{id}|_G$ for some subgroup $G \subset K_0(A)$ for which $\rho_A(G)$ is dense in $\rho_A(K_0(A))$. Then $TR(A \rtimes_\alpha \mathbb{Z}) = 0$.*

Corollary 4.6. *Let A be a unital simple AH-algebra with slow dimension growth and with real rank zero and let $\alpha \in Aut(A)$. Suppose that α has the tracial Rokhlin property and $[\alpha^r] = [\mathrm{id}_A]$ in $KL(A, A)$ for some integer $r \geq 1$. Then $A \rtimes_\alpha \mathbb{Z}$ is again an AH-algebra with slow dimension growth and with real rank zero.*

Proof. It is known that $TR(A) = 0$. By 4.5, $TR(A \rtimes_\alpha \mathbb{Z}) = 0$. $A \rtimes_\alpha \mathbb{Z}$ also satisfies the Universal Coefficient Theorem, by [20], $A \rtimes_\alpha \mathbb{Z}$ is an AH-algebra with slow dimension growth and with real rank zero. $\square$

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